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Algebraic shape invariant potentials as the generalized deformed oscillator

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Abstract

Within the framework of supersymmetric quantum mechanics, we study the simplified version of the potential algebra of the shape invariance condition in k steps, where k is an arbitrary positive integer. The associated potential algebra is found to be equivalent to the generalized deformed oscillator algebra that has a built-in Z_k-grading structure. The algebraic realization of the shape invariance condition in k steps is therefore formulated by the method of the Z_k-graded deformed oscillator. Based on this formulation, we explicitly construct the general algebraic properties for shape invariant potentials in k steps, in which the parameters of partner potentials are related to each other by the translation $a_1 = a_0 + \delta$. The obtained results include the cyclic shape invariant potentials of period k as a special case.

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1. Introduction

Supersymmetry (SUSY) is the symmetry between bosonic and fermionic degrees of freedom. The idea of SUSY was initially introduced to solve the hierarchy problem in grand unified theories. SUSY is not an exact symmetry of nature; therefore, it has to be broken at some stages. It is however difficult to determine whether SUSY is broken or not in supersymmetric quantum field theories. Supersymmetric quantum mechanics (SUSYQM) as a result becomes a testing ground to understand non-perturbative SUSY breaking [1, 2]. For a review of SUSYQM, refer to [3–5] and references therein.

In SUSYQM, one constructs the SUSY partner Hamiltonian, starting from a given one-dimensional Hamiltonian by the method of factorization [6]. The process can be used successively to generate an entire hierarchy of the isospectral Hamiltonians. Let us be more specific. Two potentials $V^{(-)}(x, a_0)$ and $V^{(+)}(x, a_0)$ are said to be SUSY partner potentials if they are related to the superpotential $W(x, a_0)$ by

$$V^{(\pm)}(x, a_0) = W^2(x, a_0) \pm W'(x, a_0), \tag{1}$$

where $W'(x, a_0) \equiv \frac{\mathrm{d}}{\mathrm{d}x} W(x, a_0)$ and a_0 is a set of parameters. In units of $\hbar = 2m = 1$, the associated SUSY partner Hamiltonians $H^{(-)}(x, a_0)$ and $H^{(+)}(x, a_0)$ have the standard forms

$$H^{(\pm)}(x, a_0) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V^{(\pm)}(x, a_0). \tag{2}$$

In the case of unbroken SUSY, a special property regarding SUSYQM is that except for a zero-energy eigenstate, the SUSY partner Hamiltonians $H^{(\pm)}(x, a_0)$ are found to be exactly isospectral. That is, if we know all the eigenstates of $H^{(-)}(x, a_0)$, we can construct all the eigenstates of $H^{(+)}(x, a_0)$, and vice versa, except for the zero-energy ground-state eigenfunction. Furthermore, the existence of the zero-energy state is completely determined by the asymptotic behavior of the superpotential $W(x, a_0)$. If we denote $W(x \to \pm\infty, a_0) = W_\pm \neq 0$,[1] then $\mathrm{sgn}(W_+) = -\mathrm{sgn}(W_-)$ indicates good SUSY, whereas $\mathrm{sgn}(W_+) = \mathrm{sgn}(W_-)$ signifies the breaking of SUSY. As a result, the Witten index acquires the topological expression: $\Delta = \frac{1}{2}[\mathrm{sgn}(W_+) - \mathrm{sgn}(W_-)]$ [4]. Here, $\mathrm{sgn}(W_\pm)$ is the sign of the asymptotic value $W_\pm$, respectively.

SUSYQM has been shown to provide a key ingredient to explore exactly solvable potentials for the Schrödinger equation in nonrelativistic quantum mechanics. In this respect, the concept of the shape invariance condition [7] in the formalism of SUSYQM becomes very critical, because it leads immediately to an integrability condition to the problem. What we mean by shape invariance is that the pair of partner potentials $V^{(\pm)}(x, a_0)$ defined in equation (1) are similar in shape but differ only up to a change of parameters and additive constants. Mathematically, the condition reads[2]

$$V^{(+)}(x, a_0) = V^{(-)}(x, a_1) + R(a_0), \tag{3}$$

where $a_1 = f(a_0)$ is a function of a_0 and the remainder $R(a_0)$ is independent of x. Now by equation (3), the entire spectrum of eigenenergies for the initial Hamiltonian $H^{(-)}(x, a_0)$ (2) can be obtained algebraically as ($n = 1, 2, 3, \ldots$) [7, 8]

$$E_0^{(-)} = 0, \qquad E_n^{(-)} = \sum_{i=0}^{n-1} R(a_i). \tag{4}$$

Here, we assume that the superpotential $W(x, a_0)$ is constructed in such a way that the Hamiltonian $H^{(-)}(x, a_0)$ possesses the unique zero-energy ground state.

Many interesting classes of solvable shape invariant potentials in one step that retain SUSY have been reported and discussed, including all the analytically solvable potentials known in the context of nonrelativistic quantum mechanics. In the SUSYQM literature, there are four typical classes: (i) the translation class [9, 10], where the parameters a_0 and a_1 are related by translation $a_1 = a_0 + \delta$; (ii) the scaling class [11, 12], where the parameters are related by $a_1 = qa_0$ and $0 < q < 1$; (iii) the cyclic class [13], where $a_0 = a_p$ and $f(a_0) = a_1 = a_{p+1}$, for $p = 2, 3, \ldots$; and (iv) the 'exotic' class [11], where $a_1 = qa_0^p$ and its generalization $a_1 = qa_0/(1 + pa_0)$, for $0 < q < 1$ and $p = 2, 3, \ldots$.[3]

To obtain more solvable shape invariant potentials, the concept of shape invariance can be extended to two and even multi-steps [11]. Based on this method, various shape invariant potentials in two or higher steps have been obtained [13–15]. The extension to shape invariance in multi-steps is rather straightforward. So, let us consider the case of the shape invariance

[1] If one or both values of $W_\pm$ vanish, more information about the asymptotic behavior of $W(x, a_0)$ is needed to determine whether SUSY is broken [4].

[2] Since the partner potentials $V^{(-)}(x, a_1)$ and $V^{(+)}(x, a_0)$ are related to each other by one relation, equation (3) is thus called the one-step shape invariance condition.

[3] Strictly speaking, these four classes can be transformed to one another by suitable reparameterizations. For example, the scaling form $a_1 = qa_0$ can be rearranged into the translation form $a_1' = a_0' + \delta$ by taking logarithms.

condition in k steps with unbroken SUSY, where k is an arbitrary positive integer. We are given k superpotentials, denoted by $W_s(x, a_0)$ with $s = 0, 1, \ldots, k-1$, whose asymptotic values are assumed to satisfy the conditions $\mathrm{sgn}(W_{0+}) = \mathrm{sgn}(W_{1+}) = \cdots = \mathrm{sgn}(W_{(k-1)+})$ and $\mathrm{sgn}(W_{s+}) = -\mathrm{sgn}(W_{s-})$ for all these superpotentials. Further, the respective partner potentials $V_s^{(\pm)}(x, a_0)$, defined in equation (1), are chosen to obey the following k relations:

$$\begin{aligned} V_0^{(+)}(x, a_0) &= V_1^{(-)}(x, a_0) + R_0(a_0), \\ V_1^{(+)}(x, a_0) &= V_2^{(-)}(x, a_0) + R_1(a_0), \\ \cdots &= \cdots \\ V_{k-1}^{(+)}(x, a_0) &= V_0^{(-)}(x, a_1) + R_{k-1}(a_0), \end{aligned} \tag{5}$$

where the k arbitrary remainders $R_s(a_0)$ are independent of x. By shape invariance condition in k steps, we mean that the SUSY partner potentials $V_0^{(-)}(x, a_1)$ and $V_0^{(+)}(x, a_0)$ at this time are related to each other by the above k relations.

It is readily shown that from equation (5) the energy eigenvalues for the initial potential $V_0^{(-)}(x, a_0)$ can be algebraically determined. These eigenvalues are found as

$$E_{nk+s}^{(-)} = \sum_{m=0}^{n-1} \sum_{t=0}^{k-1} R_t(a_m) + \sum_{t=0}^{s-1} R_t(a_n), \tag{6}$$

where $E_0^{(-)} = 0$, $s = 0, 1, \ldots, k-1$ and $n = 0, 1, 2, \ldots$. Note that we use the convention for the summation $\sum_{t=0}^{-1} = 0$.

The purpose of the present paper is to explore the possible algebraic structures of the shape invariance condition in k steps, as described by equations (5) and (6). The general solution to this problem remains unsolved. However, as we shall show, if extra relations among the k unrelated superpotentials $W_s(x, a_0)$ and among the k unrelated remainders $R_s(a_0)$ can be separately introduced, the underlying potential algebra will be simplified. This simplified version of the potential algebra is found to be equivalent to the so-called generalized deformed oscillator algebra with a built-in Z_k-grading structure. Therefore, the algebra of the shape invariance condition in k steps can be studied in the context of the Z_k-graded generalized deformed oscillator [16, 17]. For the purpose of illustration, we explicitly work out the detailed algebraic properties of shape invariant potentials in general k steps, in which the parameters of partner potentials are related to each other by a translational change $a_1 = a_0 + \delta$. These results include the cyclic shape invariant potentials of period k as a particular case [13, 18, 19]. In addition, the results also contain two new types of shape invariant potentials in k steps: the Pöschl–Teller I & II potentials in k steps that generalize the ordinary shape invariant Pöschl–Teller I & II potentials from one step to k steps, respectively.

The paper is organized as follows. In section 2, by imposing the extra relations we establish the simplified version of potential algebra of shape invariance condition in k steps. In section 3, we review and modify the basic definitions and notations of the generalized deformed oscillator with a built-in Z_k-grading structure. In section 4, we explicitly work out the algebraic properties of translational shape invariant potentials in k steps, based on the Z_k-graded deformed oscillator algebra. Finally, section 5 contains a discussion.

2. Algebraic shape invariance in k steps

It is known that the shape invariance condition in one step (3) possesses what is generally referred to as a potential algebra [20, 21]. That is, the one-step shape invariant potentials

have an underlying algebraic structure, and thus can be studied by group theoretical methods [22, 23]. As a result, the energy eigenvalues of the one-step shape invariant Hamiltonian can be determined by purely algebraic means.

With some restriction, the concept of potential algebra can also be applied to the shape invariance condition in more than one step. Recently, a simplified version of potential algebra for the shape invariance condition in two steps has been constructed and discussed by the author [24]. It is found that the corresponding simplified potential algebra is similar to that of shape invariance in one step, and is based only on the three angular-momentum-like generators. It is also suggested that the same technique can be further extended to the shape invariance condition in more than two steps. In this section, we shall continue to investigate the general algebraic properties for the shape invariance condition in multi-steps in a concrete and consistent way.

To begin with, let us consider in equation (5) the substitution of parameters $a_0 \to \boldsymbol{\alpha}(N_0)$ and for the general case the substitution $a_m \to \boldsymbol{\alpha}(N_0 - m)$, where N_0 is an arbitrary positive integer and $m = 0, 1, 2, \ldots$. The precise form of the function $\boldsymbol{\alpha}(N_0)$ is to be determined by requiring that the change $\boldsymbol{\alpha}(N_0) \to \boldsymbol{\alpha}(N_0 - 1)$ correspond to the change of parameters $a_0 \to a_1$. In terms of the k superpotentials, $W_s(x, \boldsymbol{\alpha}(N_0))$ with $s = 0, 1, \ldots, k-1$, the corresponding shape invariance condition in k steps (5) can be expressed as

$$\begin{aligned}
&W_0^2(\boldsymbol{\alpha}(N_0)) + W_0'(\boldsymbol{\alpha}(N_0)) = W_1^2(\boldsymbol{\alpha}(N_0)) - W_1'(\boldsymbol{\alpha}(N_0)) + R_0(\boldsymbol{\alpha}(N_0)),\\
&W_1^2(\boldsymbol{\alpha}(N_0)) + W_1'(\boldsymbol{\alpha}(N_0)) = W_2^2(\boldsymbol{\alpha}(N_0)) - W_2'(\boldsymbol{\alpha}(N_0)) + R_1(\boldsymbol{\alpha}(N_0)),\\
&\cdots = \cdots\\
&W_{k-1}^2(\boldsymbol{\alpha}(N_0)) + W_{k-1}'(\boldsymbol{\alpha}(N_0)) = W_0^2(\boldsymbol{\alpha}(N_0 - 1)) - W_0'(\boldsymbol{\alpha}(N_0 - 1)) + R_{k-1}(\boldsymbol{\alpha}(N_0)).
\end{aligned} \tag{7}$$

To simplify the notation, we have suppressed the x-dependence in the superpotentials $W_s(x, \boldsymbol{\alpha}(N_0))$ in equation (7). These k superpotentials are quite arbitrary at this stage. Without further clues, obviously it is very difficult to determine the underlying potential algebra described by the above k relations.

To solve this problem, we restrict ourselves to a particular subset of the shape invariance condition in k steps, instead. The subset is obtained by imposing extra relations on the k superpotentials and on the k remainders. This will result in a simplified version of the corresponding potential algebra. Explicitly, these relations are based on the following particular identification:

$$W_s(x, \boldsymbol{\alpha}(N_0)) \equiv W\left(x, \boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right), \qquad R_s(\boldsymbol{\alpha}(N_0)) \equiv R\left(\boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right). \tag{8}$$

With the help of equation (8), the k seemly unrelated relations in (7) can be cast into a single and compact equation for the unified superpotential $W(x, \boldsymbol{\alpha}(N_0))$ and the unified remainder $R(\boldsymbol{\alpha}(N_0))$ as

$$\begin{aligned}
W^2\left(x, \boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right) + W'\left(x, \boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right) &= W^2\left(x, \boldsymbol{\alpha}\left(N_0 - \frac{s+1}{k}\right)\right)\\
&\quad - W'\left(x, \boldsymbol{\alpha}\left(N_0 - \frac{s+1}{k}\right)\right) + R\left(\boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right).
\end{aligned} \tag{9}$$

We note that the k relations in equation (7) can be easily reproduced from equation (9) by letting, one at a time, $s = 0, 1, \ldots, k-1$, respectively.

Equation (9) actually represents a constraint equation for the superpotential $W(x, \boldsymbol{\alpha}(N_0))$ when the parameter N_0 is changed by $-\frac{1}{k}$ as we go from the first superpotential to the second one, third one, fourth one, and so on. From the viewpoint of quantum mechanics, this change of parameter, i.e. $N_0 \to N_0 - \frac{1}{k}$, can be formally accomplished by the action of the raising and

lowering operators of angular momentum [22], or equivalently, by the action of the creation and annihilation operators of a simple harmonic oscillator. Here, we shall adopt the notion of the simple harmonic oscillator in the description and treatment of our problem. With this in mind, we first define an operator $\mathcal{N}$ that is analogous to the number operator of the harmonic oscillator by

$$\mathcal{N} \equiv \frac{1}{\mathrm{i}} \frac{\partial}{\partial \phi}, \tag{10}$$

and designate the parameter N_0 as the eigenvalue of the number operator $\mathcal{N}$ acting on the eigenstate $|N_0\rangle$, that is $\mathcal{N}|N_0\rangle = N_0|N_0\rangle$. Now, it is natural to think that the unified superpotential function lives in a two-dimensional space spanned by the coordinates x and ϕ, and the eigenvalue of $\mathcal{N}$ is assigned to the labeling parameter. In this way, the set of k relations in equation (7) can be immediately generated when we project this operator equation

$$W^2\left(x, \boldsymbol{\alpha}\left(\mathcal{N}+\frac{1}{k}\right)\right) + W'\left(x, \boldsymbol{\alpha}\left(\mathcal{N}+\frac{1}{k}\right)\right) = W^2(x, \boldsymbol{\alpha}(\mathcal{N})) - W'(x, \boldsymbol{\alpha}(\mathcal{N})) + R\left(\boldsymbol{\alpha}\left(\mathcal{N}+\frac{1}{k}\right)\right) \tag{11}$$

into the eigenstates $\left|N_0 - \frac{s+1}{k}\right\rangle$, for $s = 0, 1, \ldots, k-1$, respectively.

Next, we construct the annihilation and creation operators of the associated harmonic oscillator. The annihilation operator $\mathcal{A}$ and creation operator $\mathcal{A}^\dagger = (\mathcal{A})^\dagger$ are built using the unified superpotential $W(x, \alpha(\mathcal{N}))$ as

$$\mathcal{A} = \mathrm{e}^{-\mathrm{i}\phi/k}\left[\frac{\partial}{\partial x} + W(x, \boldsymbol{\alpha}(\mathcal{N}))\right], \qquad \mathcal{A}^\dagger = \left[-\frac{\partial}{\partial x} + W(x, \boldsymbol{\alpha}(\mathcal{N}))\right]\mathrm{e}^{\mathrm{i}\phi/k}. \tag{12}$$

Explicit computations then show that the simplified potential algebra of shape invariance in k steps, based on the set of generators $\{I, \mathcal{A}^\dagger, \mathcal{A}, \mathcal{N}\}$, is indeed closed and generically nonlinear. Their commutation relations are described by

$$[\mathcal{A}, \mathcal{N}] = \frac{1}{k}\mathcal{A}, \qquad [\mathcal{A}^\dagger, \mathcal{N}] = -\frac{1}{k}\mathcal{A}^\dagger, \qquad [\mathcal{A}^\dagger, \mathcal{A}] = -R\left(\boldsymbol{\alpha}\left(\mathcal{N}+\frac{1}{k}\right)\right), \tag{13}$$

where we have used equation (11). In this way, the operator $\mathcal{A}^\dagger(\mathcal{A})$, as desired, changes the eigenvalues of $\mathcal{N}$ by $+(-)\frac{1}{k}$, respectively. We mention here that the above potential algebra (13) based on the set of generators $\{I, \mathcal{A}^\dagger, \mathcal{A}, \mathcal{N}\}$ is similar to that of shape invariance condition in one step [20–23]. Moreover, based on our formulation, this algebra is also equivalent to the so-called generalized deformed oscillator algebra, which has been extensively studied [26–29]. In other words, the particular identification (8) not only reduces the shape invariance condition effectively from k steps to one step, but also simplifies the associated potential algebra (7) to the well-established deformed oscillator algebra.

It is noted that the configuration space for the variable ϕ is $[0, 2\pi k]$ due to the requirement that the operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ in equation (12) remain intact under the transformation $\phi \to \phi + 2\pi k$. However, the configuration space for the variable x is not entirely specified since it depends on the singular property of the superpotential $W(x, \boldsymbol{\alpha}(\mathcal{N}))$ at the origin $x = 0$. For instance, if the singularity is not present or if it is the so-called 'soft' singularity [25] that can be properly regularized [14], the configuration space for x can be defined on either $x \in \mathbb{R}$ or $x \in [-l, l]$, where l is a constant. Otherwise, it can be only defined on either $x \in \mathbb{R}^+$ or $x \in [0, l]$. Here, based on a similar study for the shape invariant potentials in two steps [24], we shall assume that the singularity present in the unified superpotential $W(x, \boldsymbol{\alpha}(\mathcal{N}))$ is at most 'soft'.

The representation of the simplified potential algebra (13) can be obtained as follows. We consider the simultaneous eigenstates, denoted by $\left|\frac{n}{k}\right\rangle$ for $n = 0, 1, 2, \ldots$, of the mutually commuting operators $\mathcal{A}^\dagger\mathcal{A}$ and $\mathcal{N}$, whose eigenvalue equations are given by

$$\mathcal{A}^\dagger\mathcal{A}\left|\frac{n}{k}\right\rangle = \mathcal{F}\left(\boldsymbol{\alpha}\left(\frac{n}{k}\right)\right)\left|\frac{n}{k}\right\rangle, \qquad \mathcal{N}\left|\frac{n}{k}\right\rangle = \frac{n}{k}\left|\frac{n}{k}\right\rangle, \tag{14}$$

respectively. Here, we shall call $\mathcal{F}\left(\boldsymbol{\alpha}\left(\frac{n}{k}\right)\right)$ the structure function as suggested in the generalized deformed oscillator algebra. Both equations together imply the action of ladder-type operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ on the eigenstate $\left|\frac{n}{k}\right\rangle$ as

$$\mathcal{A}\left|\frac{n}{k}\right\rangle = \sqrt{\mathcal{F}\left(\boldsymbol{\alpha}\left(\frac{n}{k}\right)\right)}\left|\frac{n-1}{k}\right\rangle, \qquad \mathcal{A}^\dagger\left|\frac{n}{k}\right\rangle = \sqrt{\mathcal{F}\left(\boldsymbol{\alpha}\left(\frac{n+1}{k}\right)\right)}\left|\frac{n+1}{k}\right\rangle, \tag{15}$$

where without loss of generality the structure function $\mathcal{F}\left(\boldsymbol{\alpha}\left(\frac{n}{k}\right)\right)$ is chosen to be real and positive. If the spectrum of the operators $\mathcal{A}^\dagger\mathcal{A}$ exhibits a lowest-weight eigenstate, that is $\mathcal{A}|0\rangle = 0$, we then choose the condition $\mathcal{F}(\boldsymbol{\alpha}(0)) = 0$ to be satisfied.

The energy eigenvalues of the initial Hamiltonian $H_0^{(-)}(x, a_0) = H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$ can be expressed purely in terms of the structure function $\mathcal{F}\left(\boldsymbol{\alpha}\left(\frac{n}{k}\right)\right)$. We determine this relation by simply projecting the third commutator of equation (13) on the eigenstate $\left|N_0 - \frac{s+1}{k}\right\rangle$, and obtain

$$\mathcal{F}\left(\boldsymbol{\alpha}\left(N_0 - \frac{s+1}{k}\right)\right) - \mathcal{F}\left(\boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right) = -R\left(\boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right). \tag{16}$$

Then applying equation (16) recursively, we establish from equation (6) the eigenenergies of the initial Hamiltonian $H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$ (2) as

$$E_n^{(-)} = \sum_{s=0}^{n-1} R\left(\boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right) = \mathcal{F}(\boldsymbol{\alpha}(N_0)) - \mathcal{F}\left(\boldsymbol{\alpha}\left(N_0 - \frac{n}{k}\right)\right), \tag{17}$$

where $n = 0, 1, 2, \ldots$. Equation (17) describes the energy spectrum of the simplified shape invariance condition in k steps, as compared with the more complicated energy spectrum given in equation (6).

In the present research, we are primarily interested in the algebraic structures of shape invariant potentials in k steps, in which the parameters of partner potentials are related to each other by the translation: $a_1 = a_0 + \delta$. The analytical properties of such translational shape invariant potentials in k steps are little known, except for those in two steps [14, 15]. Nevertheless, for the purpose of obtaining their algebraic properties, we choose the k remainders $R_s(a_m)$ in equation (5) having the following particular forms ($m = 0, 1, 2, \ldots$):

$$R_s(a_m) = \sigma_s + a_m\omega_s, \tag{18}$$

where σ_s and ω_s, for $s = 0, 1, \ldots, k-1$, are arbitrary constants. In fact, the choice for the k-step remainders is not unique. There is another possibility for the remainders, which results in a quadratic polynomial of the parameter a_m [15]. However, we will not pursue this more difficult problem here.

Now, as suggested by the identification (8), the simplified potential algebra associated with shape invariance in k steps can be established if we identify the k unrelated remainders in equation (18) as

$$R_s(a_m) = R_s(\boldsymbol{\alpha}(N_0 - m)) \equiv R\left(\boldsymbol{\alpha}\left(N_0 - m - \frac{s}{k}\right)\right). \tag{19}$$

By comparing both equations (18) and (19), we immediately find that the unified remainder $R\left(\boldsymbol{\alpha}\left(N_0 - m - \frac{s}{k}\right)\right)$, having the required functional property, can be determined if the k pairs of constants (σ_s, ω_s) are related to one another by

$$\frac{\sigma_s}{\omega_s} = \frac{\sigma_0}{\omega_0} + \frac{s}{k}\delta. \tag{20}$$

In this way, the unified remainder takes the form

$$R\left(\boldsymbol{\alpha}\left(N_0 - m - \frac{s}{k}\right)\right) = \left[\frac{\sigma_0}{\omega_0} + a_0 + N_0\delta - \left(N_0 - m - \frac{s}{k}\right)\delta\right]\omega_s, \tag{21}$$

where the translational change of the parameters $a_m = a_0 + m\delta$ is implied.

Two remarks are in order at this stage. (i) Inspecting the energy spectrum of the initial Hamiltonian $H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$ in equation (17), we learn that the remainder $R\left(\boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right)$ is nothing but the energy gap between two adjacent eigenstates. Therefore, we must set $R\left(\boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right) > 0$, in order to prevent energy level crossing. When letting $\left(\frac{\sigma_0}{\omega_0} + a_0\right)$, $\omega_s > 0$ and examining the explicit form of the unified remainder (21), we conclude that the system will possess an infinite number of bound states if $\delta \geqslant 0$. Otherwise, it only contains a finite number of bound states for $\delta < 0$ because equation (21) may become negative for a large enough value of $\left(m + \frac{s}{k}\right)$. (ii) The particular form of the unified remainder in equation (21) actually imposes an extra Z_k-grading structure into the energy spectrum of shape invariant potentials in k steps. This can be easily checked by noting that the unified remainder in equation (21) is decomposed into k distinct combinations specified by ω_s, for each different choice of $s (s = 0, 1, \ldots, k-1)$. Because of the choice of the remainders (18), the simplified potential algebra of shape invariance in k steps is further found to be equivalent to the generalized deformed oscillator algebra that has a built-in Z_k-grading structure. The algebraic properties of shape invariant potentials in k steps can therefore be formulated by the method of Z_k-graded generalized deformed oscillator. The algebra of the Z_k-graded deformed oscillator is reviewed and modified in the next section.

3. Z_k-graded generalized deformed oscillator

For the purpose of establishing the definitions and notations, we shall begin with a brief review on the basic principles of the generalized deformed oscillator algebra. Then we introduce the generalized deformed oscillator algebra with a Z_k-grading structure. In order for our presentation to be consistent with the formulations already described in the preceding section, we will make some modifications on the Z_k-graded deformed oscillator algebra.

3.1. Generalized deformed oscillator algebra

Deformed oscillators have been proposed and studied in many different deformation schemes [26–29]. All deformed oscillators can be unified into a common mathematical framework in the formulation of the so-called generalized deformed oscillator [26]. A generalized deformed oscillator is defined by a nonlinear algebra generated by the operators I, a, $a^\dagger$, and N that fulfil the Hermiticity conditions $(a)^\dagger = a^\dagger$, $N^\dagger = N$, and the relations

$$[a, N] = a, \qquad [a^\dagger, N] = -a^\dagger, \qquad a^\dagger a = F(N), \qquad aa^\dagger = F(N+1), \tag{22}$$

where N is the number operator and the Hermitian nonnegative function $F(N)$ is the structure function. In order to have the Fock representation, $F(N)$ should satisfy the condition $F(0) = 0$ and $F(n) > 0$, for $n = 1, 2, 3, \ldots$. Technically, the structure function $F(N)$ is the characteristic of the deformation scheme. Different structure functions correspond to different

deformed oscillators. Nevertheless, all the deformed oscillators can be accommodated within the framework as given in equation (22).

To realize the algebra of the generalized deformed oscillator, it is natural to introduce the Fock space of eigenstates of the number operator N, which have the property $N|n\rangle = n|n\rangle$ and $\langle n|m\rangle = \delta_{n,m}$ (for $n, m = 0, 1, 2, \ldots$). If the ground state satisfies the relation $a|0\rangle = 0$, the complete number eigenstates are obtained by successive application of the operator $a^\dagger$ as

$$|n\rangle = \frac{1}{\sqrt{F(n)!}}(a^\dagger)^n|0\rangle, \tag{23}$$

where $n = 0, 1, \ldots, d-1$ and d may be finite or infinite. The normalization coefficients $F(n)!$ are given by

$$F(n)! = \prod_{k=1}^{n} F(k), \qquad F(0)! = 1. \tag{24}$$

In this sense, the operators a and $a^\dagger$ are the annihilation and creation operators of this deformed oscillator algebra,

$$a|n\rangle = \sqrt{F(n)}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{F(n+1)}|n+1\rangle. \tag{25}$$

3.2. Z_k-graded deformed oscillator algebra

We introduce the Z_k-graded deformed oscillator algebra that possesses a built-in Z_k-grading structure in the subsection. In the literature, there are various versions of the Z_k-extended deformed oscillator algebra that have been discussed [16, 17]. Here, we make some modifications on this Z_k-extended algebra for the purpose of incorporating the algebraic properties of the k-step shape invariance condition.

To be more specific, the algebra of the Z_k-graded generalized deformed oscillator is defined by the operators I, a, $a^\dagger$, N and T that fulfil the Hermiticity conditions $(a)^\dagger = a^\dagger$, $N^\dagger = N$, $T^\dagger = T^{-1}$ and the following relations:

$$[a, N] = \frac{1}{k}a, \qquad [a^\dagger, N] = -\frac{1}{k}a^\dagger, \qquad a^\dagger a = F(N), \qquad aa^\dagger = F\left(N + \frac{1}{k}\right), \tag{26}$$

$$T^k = I, \qquad [N, T] = 0 = [N, T^\dagger], \qquad a^\dagger T = \mathrm{e}^{-\mathrm{i}2\pi/k}Ta^\dagger, \qquad T^\dagger a = aT^\dagger\,\mathrm{e}^{\mathrm{i}2\pi/k}, \tag{27}$$

where $k = 1, 2, 3, \ldots$. For the special value $k = 1$, it is readily known that the algebra defined in equations (26) and (27) reduces to the generalized deformed oscillator algebra (22). In addition, we note that in defining the Z_k-graded deformed oscillator algebra above, the creation operator $a^\dagger$ is designated to increase the eigenvalue of the number operator N by units of $\frac{1}{k}$, not by unity as in the conventional Z_k-extended deformed oscillator. Similarly, the annihilation operator a decreases it by $\frac{1}{k}$.

The grading operator T in equation (27) is the generator of the cyclic group of order k, $Z_k = \{1, T, T^2, \ldots, T^{k-1} | T^k = 1\}$. There can be many different realizations for such a operator, but here we choose the one in which the grading operator is easily realized in terms of the number operator N as

$$T = \mathrm{e}^{2\pi\mathrm{i}N}. \tag{28}$$

The grading operator T has k distinct eigenvalues q^s in which q is the sth root of unity and $s = 0, 1, 2, \ldots, k-1$. Therefore, the Z_k-grading structure of the Fock space $\mathcal{H}$ can be

distinguished by the eigenvalues of the grading operator T, for which the corresponding Fock subspace of eigenvalue q^s is denoted by

$$\mathcal{H}_s : \left\{ \left| n + \frac{s}{k} \right\rangle \middle| n = 0, 1, 2, \ldots \right\}. \tag{29}$$

The entire Fock space is consequently the direct sum of each individual Fock subspace as given by $\mathcal{H} = \sum_{s=0}^{k-1} \oplus \mathcal{H}_s$.

We have shown that the Fock space of the Z_k-graded deformed oscillator consists of k distinct Fock subspaces, which can be specified by the grading operator T. In fact, the grading structure of the Fock space can also be naturally formulated by the projection operators that, by construction, project on these distinct subspaces. Explicitly, the projection operators are expressed in terms of the grading operator T as (for $s = 0, 1, \ldots, k-1$)

$$\Pi_s = \frac{1}{k} \sum_{t=0}^{k-1} \mathrm{e}^{-2\pi \mathrm{i} t s/k} T^t = \frac{1}{k} \sum_{t=0}^{k-1} \mathrm{e}^{2\pi \mathrm{i} t (N - s/k)}, \qquad \sum_{s=0}^{k-1} \Pi_s = I. \tag{30}$$

At this stage, we are able to restate the algebra of the Z_k-graded generalized deformed oscillator, using the projection operator Π_s not the grading one T: the algebra of the Z_k-graded deformed oscillator is generated by the set of operators I, a, $a^\dagger$, N and Π_s, that fulfil the Hermiticity conditions $(a)^\dagger = a^\dagger$, $N^\dagger = N$, $\Pi_s^\dagger = \Pi_s$ and the following relations:

$$[a, N] = \frac{1}{k} a, \qquad [a^\dagger, N] = -\frac{1}{k} a^\dagger, \qquad a^\dagger a = F(N), \qquad a a^\dagger = F\left(N + \frac{1}{k}\right), \tag{31}$$

$$[N, \Pi_s] = 0, \qquad \Pi_s \Pi_t = \delta_{s,t}, \qquad a^\dagger \Pi_s = \Pi_{s+1} a^\dagger, \qquad a \Pi_s = \Pi_{s-1} a, \tag{32}$$

where $s, t = 0, 1, \ldots, k-1$. The convention for the projection operators has been used: $\Pi_t = \Pi_s$ if $t - s = 0 \bmod k$.

To establish the complete Fock space representation for the Z_k-graded deformed oscillator, let us again use the creation and annihilation operators $a^\dagger$ and a, and take the eigenstates of the number operator N, which fulfil the properties $N\left|\frac{n}{k}\right\rangle = \frac{n}{k}\left|\frac{n}{k}\right\rangle$ and $\left\langle \frac{n}{k} \middle| \frac{m}{k} \right\rangle = \delta_{n,m}$, for $n, m = 0, 1, 2, \ldots$. The ground state is defined by the simultaneous eigenstate of the operators N and $a^\dagger a$ with both vanishing eigenvalues, and in addition satisfies the relation $a|0\rangle = 0$. The complete Fock-space eigenstates are constructed by operator $a^\dagger$ as

$$\left|\frac{n}{k}\right\rangle = \frac{1}{\sqrt{F\left(\frac{n}{k}\right)!}} (a^\dagger)^n |0\rangle, \tag{33}$$

where the normalization coefficients are

$$F\left(\frac{n}{k}\right)! = \prod_{m=1}^{n} F\left(\frac{m}{k}\right), \qquad F(0)! = 1. \tag{34}$$

In this way, the eigenvalue equations for the operators N and Π_t are

$$N \left| n + \frac{s}{k} \right\rangle = \left(n + \frac{s}{k}\right) \left| n + \frac{s}{k} \right\rangle, \tag{35}$$

$$\Pi_t \left| n + \frac{s}{k} \right\rangle = \delta_{t,s} \left| n + \frac{s}{k} \right\rangle, \tag{36}$$

where $s, t = 0, 1, \ldots, k-1$. Moreover, the operators a and $a^\dagger$ act on these eigenstates, rendering

$$a \left| n + \frac{s}{k} \right\rangle = \sqrt{F\left(n + \frac{s}{k}\right)} \left| n + \frac{s-1}{k} \right\rangle, \tag{37}$$

$$a^{\dagger}\left|n+\frac{s}{k}\right\rangle=\sqrt{F\left(n+\frac{s+1}{k}\right)}\left|n+\frac{s+1}{k}\right\rangle. \tag{38}$$

Finally, because of the encoded Z_k-grading structure, the corresponding structure function $F(N)$, expressed in terms of the projection operators Π_s, will have the following form:

$$F(N)=f(N)+\sum_{s=0}^{k-1}g_s(N)\Pi_s. \tag{39}$$

Here, $f(N)$ and $g_s(N)$ are analytical functions of N.

4. Algebraic structures of translational shape invariant potentials

To illustrate the simplified potential algebra developed in section 2, we shall investigate algebraic structures for the shape invariant potentials in k steps, in which the parameters of partner potentials are related to each other by a translation change: $a_m = a_{m-1}+\delta = a_0+m\delta$. The quantity δ is a constant and $m = 0, 1, 2, \ldots$. The analysis will be based on the formalism of the Z_k-graded deformed oscillator reviewed and modified in section 3. The relevant algebraic quantities for the shape invariant potentials in the first few number of k steps will be explicitly constructed. After obtaining enough information for these potentials, the general algebraic results for the shape invariant potentials in arbitrary k steps can be deduced.

Before presenting the details, we mention here some important relations that are used in the upcoming calculations. We note that the unified remainder given in equation (21) can be expressed in terms of the number operator $\mathcal{N}$ and projection operators Π_s. It takes the compact form as

$$R(\boldsymbol{\alpha}(\mathcal{N}))=(\mathcal{C}+(N_0-\mathcal{N})\delta)\sum_{s=0}^{k-1}\omega_s\Pi_s, \tag{40}$$

where $\mathcal{C}$ is a short-hand notation,

$$\mathcal{C}=\frac{\sigma_0}{\omega_0}+a_0. \tag{41}$$

In the same vein, using the operators $\mathcal{N}$, Π_s, and equation (39), we are able to write the structure function $\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N}))$ of shape invariance condition in k steps defined in equation (14) as

$$\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N}))=f(\mathcal{N})+\sum_{s=0}^{k-1}g_s(\mathcal{N})\Pi_s, \tag{42}$$

where $f(\mathcal{N})$ and $g_s(\mathcal{N})$ are functions of $\mathcal{N}$. All of them are determined by requiring equation (42) to satisfy the remainder-structure function relation (16), that is

$$\mathcal{F}\left(\boldsymbol{\alpha}\left(\mathcal{N}-\frac{1}{k}\right)\right)-\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N}))=-R(\boldsymbol{\alpha}(\mathcal{N})). \tag{43}$$

Furthermore, we introduce two quantities that will be useful in the later presentations. The first one is the analogous Kronecker delta for the cyclic group of order k. We use the symbol $\Delta_{s,t}$ that is defined by

$$\Delta_{s,t}=\begin{cases}1, & \text{for} \quad s=t \bmod k,\\ 0, & \text{for} \quad s\neq t \bmod k.\end{cases} \tag{44}$$

The second one is the quantity Ω_k given by the summation

$$\Omega_k\equiv\sum_{s=0}^{k-1}\omega_s. \tag{45}$$

4.1. Shape invariance in two steps

Let us begin with the simplest example, which is the shape invariance condition in two steps. For $k = 2$, the grading operator $\exp(2\pi \mathrm{i}\mathcal{N})$ simply reduces to the usual Klein operator. The corresponding projection operators $\Pi_0 = \frac{1}{2}(I + (-1)^{2\mathcal{N}})$ and $\Pi_1 = \frac{1}{2}(I - (-1)^{2\mathcal{N}})$ project upon the even subspace $\mathcal{H}_0 = \{|N_0 - n\rangle | n = 0, 1, 2, \ldots\}$ and odd subspace $\mathcal{H}_1 = \left\{\left|N_0 - n - \frac{1}{2}\right\rangle | n = 0, 1, 2, \ldots\right\}$ of the Z_2-graded Fock space $\mathcal{H}$, respectively. In this sense, the Z_2-graded deformed oscillator algebra is similar to the Calogero–Vasiliev oscillator algebra [30]. Moreover, it is also equivalent to the so-called R-deformed Heisenberg algebra [27] which has found many interesting applications recently [31].

When applying the operator equation (40) on the two eigenstates $|N_0\rangle$ and $\left|N_0 - \frac{1}{2}\right\rangle$, we obtain the first two remainders for the initial Hamiltonian $H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$ as $R(\boldsymbol{\alpha}(N_0)) = \mathcal{C}\omega_0$ and $R\left(\boldsymbol{\alpha}\left(N_0 - \frac{1}{2}\right)\right) = \left(\mathcal{C} + \frac{1}{2}\delta\right)\omega_1$, respectively. The values of $R\left(\boldsymbol{\alpha}\left(N_0 - \frac{1}{2}n\right)\right)$ for other number eigenstates can be obtained in the similar manner. Using these results, the corresponding structure function $\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N}))$ for the shape invariant potentials in two steps can be determined by equation (43). After some calculations, we arrive at

$$\mathcal{F}\left(\boldsymbol{\alpha}\left(N_0 - \frac{1}{2}n\right)\right) = \mathcal{C}_0 - \frac{1}{2}\Omega_2\left(\mathcal{C} + \frac{1}{4}(n-1)\delta\right)n + \sum_{s=0}^{1}\left(\left(\mathcal{C} + \frac{1}{4}(2n-1)\delta\right)c_s - \delta d_s\right)\Delta_{s,n}, \tag{46}$$

where $n = 0, 1, 2, \ldots$ and $\mathcal{C}_0$ is an arbitrary constant to render $\mathcal{F}\left(\boldsymbol{\alpha}\left(N_0 - \frac{n}{2}\right)\right)$ positive definite. Due to the presence of $\Delta_{s,n}$ (44), only one term in the summation is singled out which fulfils the condition $n - s = 0 \bmod 2$. Moreover, the constants c_s and d_s ($s = 0, 1$) are given by

$$c_0 = -c_1 = \tfrac{1}{4}(\omega_0 - \omega_1), \qquad d_0 = d_1 = \tfrac{1}{8}(\omega_0 + \omega_1). \tag{47}$$

The energy spectrum (17) of the initial Hamiltonian $H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$ is therefore

$$E_n^{(-)} = \mathcal{F}(\boldsymbol{\alpha}(N_0)) - \mathcal{F}\left(\boldsymbol{\alpha}\left(N_0 - \tfrac{1}{2}n\right)\right). \tag{48}$$

We note that a similar algebraic result for shape invariant potentials in two steps has been constructed using a different approach [24].

For the purpose of completeness, we review the relevant parts for the shape invariant potentials in two steps. The detailed analytical properties can be found in [14, 15]. For brevity, the unified superpotential takes the simple form

$$W(x, \boldsymbol{\alpha}(\mathcal{N})) = \left(\mathcal{C} + \left(N_0 - \mathcal{N} - \frac{1}{4}\right)\delta\right)g(x) + \frac{1}{2g(x)}\sum_{s=0}^{1} c_s \Pi_s, \tag{49}$$

where the function $g(x)$ is determined by the first-order differential equation

$$g'(x) - \frac{\delta}{2}g^2(x) = \frac{1}{4}(\omega_0 + \omega_1). \tag{50}$$

Consequently, the shape invariant potentials in two steps are: (i) for $\delta = 0$, the singular harmonic oscillator potential, (ii) for $\delta > 0$, the singular Pöschl–Teller I potential and (iii) for $\delta < 0$, the singular Pöschl–Teller II potential. For each of these three potentials, the superpotential is continuous with well-defined derivatives everywhere, except at the origin $x = 0$, where it is singular and has an infinite discontinuity. A regularization that preserves shape invariance needs to be introduced, hence giving rise to the corresponding regularized potential a Dirac delta-function singularity at the origin [14].

4.2. Shape invariance in three steps

The next allowed value is $k = 3$, that is the potential algebra of the shape invariance condition in three steps. The grading operator is again denoted by $\exp(2\pi \mathrm{i}\mathcal{N})$. The Z_3-graded Fock space is constructed by the direct sum: $\mathcal{H} = \sum_{s=0}^{2} \oplus \mathcal{H}_s$. The projection operators are given by

$$\begin{aligned} \Pi_0 &= \tfrac{1}{3}(1 + 2\cos 2\pi\mathcal{N}), \\ \Pi_1 &= \tfrac{1}{3}(1 - \cos 2\pi\mathcal{N} - \sqrt{3}\sin 2\pi\mathcal{N}), \\ \Pi_2 &= \tfrac{1}{3}(1 - \cos 2\pi\mathcal{N} + \sqrt{3}\sin 2\pi\mathcal{N}), \end{aligned} \tag{51}$$

that project on the Fock subspace $\mathcal{H}_s = \left\{ \left| N_0 - n - \frac{s}{3} \right\rangle | n = 0, 1, 2, \ldots \right\}$, for $s = 0$, 1, and 2, respectively.

It is readily checked that when acting equation (40) on the first three eigenstates $|N_0\rangle$, $\left|N_0 - \frac{1}{3}\right\rangle$ and $\left|N_0 - \frac{2}{3}\right\rangle$, we obtain $R(\boldsymbol{\alpha}(N_0)) = \mathcal{C}\omega_0$, $R\left(\boldsymbol{\alpha}\left(N_0 - \frac{1}{3}\right)\right) = \left(\mathcal{C} + \frac{1}{3}\delta\right)\omega_1$ and $R\left(\boldsymbol{\alpha}\left(N_0 - \frac{2}{3}\right)\right) = \left(\mathcal{C} + \frac{2}{3}\delta\right)\omega_2$, respectively. Other results can be similarly obtained. By equation (43), we establish the corresponding structure function after some calculations ($n = 0, 1, 2, \ldots$):

$$\begin{aligned} \mathcal{F}\left(\boldsymbol{\alpha}\left(N_0 - \frac{1}{3}n\right)\right) &= \mathcal{C}_0 - \frac{1}{3}\Omega_3\left(\mathcal{C} + \frac{1}{6}(n-1)\delta\right)n \\ &\quad + \sum_{s=0}^{2}\left(\left(\mathcal{C} + \frac{1}{6}(2n-1)\delta\right)c_s - \delta d_s\right)\Delta_{s,n}, \end{aligned} \tag{52}$$

where, as in the previous case, only one term is singled out in the summation that satisfies the condition $n - s = 0 \bmod 3$. The constant $\mathcal{C}_0$ is chosen to make the associated structure function positive definite. The constants c_s and d_s for $s = 0$, 1, 2 in equation (52) are found to be

$$\begin{aligned} c_0 &= \tfrac{1}{3}(\omega_0 - \omega_2), & d_0 &= \tfrac{1}{18}(2\omega_0 + \omega_1 + 2\omega_2), \\ c_1 &= \tfrac{1}{3}(\omega_1 - \omega_0), & d_1 &= \tfrac{1}{18}(2\omega_1 + \omega_2 + 2\omega_0), \\ c_2 &= \tfrac{1}{3}(\omega_2 - \omega_1), & d_2 &= \tfrac{1}{18}(2\omega_2 + \omega_0 + 2\omega_1). \end{aligned} \tag{53}$$

The energy spectrum for the shape invariant potentials in three steps can be algebraically determined from equations (52) and (17), in which k is replaced by 3.

4.3. Shape invariance in four steps

The grading operator for shape invariant potentials is $\exp(2\pi \mathrm{i}\mathcal{N})$ once more. The Z_4-graded Fock space of the shape invariance condition in four steps is denoted by $\mathcal{H} = \sum_{s=0}^{3} \oplus \mathcal{H}_s$. The projection operators are

$$\begin{aligned} \Pi_0 &= \tfrac{1}{4}(1 + 2\cos 2\pi\mathcal{N} + \cos 4\pi\mathcal{N}), \\ \Pi_1 &= \tfrac{1}{4}(1 - 2\sin 2\pi\mathcal{N} - \cos 4\pi\mathcal{N}), \\ \Pi_2 &= \tfrac{1}{4}(1 - 2\cos 2\pi\mathcal{N} + \cos 4\pi\mathcal{N}), \\ \Pi_3 &= \tfrac{1}{4}(1 + 2\sin 2\pi\mathcal{N} - \cos 4\pi\mathcal{N}), \end{aligned} \tag{54}$$

projecting on the Fock subspace $\mathcal{H}_s = \left\{ \left| N_0 - n - \frac{s}{4} \right\rangle | n = 0, 1, 2, \ldots \right\}$, for $s = 0$, 1, 2, and 3, respectively. When acting equation (40) on the first four eigenstates of the system $|N_0\rangle$, $\left|N_0 - \frac{1}{4}\right\rangle$, $\left|N_0 - \frac{2}{4}\right\rangle$ and $\left|N_0 - \frac{3}{4}\right\rangle$, we find that $R(\boldsymbol{\alpha}(N_0)) = \mathcal{C}\omega_0$, $R\left(\boldsymbol{\alpha}\left(N_0 - \frac{1}{4}\right)\right) = \left(\mathcal{C} + \frac{1}{4}\delta\right)\omega_1$,

$R\big(\boldsymbol{\alpha}\big(N_0-\frac{2}{4}\big)\big)=\big(\mathcal{C}+\frac{2}{4}\delta\big)\omega_2$ and $R\big(\boldsymbol{\alpha}\big(N_0-\frac{3}{4}\big)\big)=\big(\mathcal{C}+\frac{3}{4}\delta\big)\omega_3$, respectively. Other values of $R\big(\boldsymbol{\alpha}\big(N_0-\frac{1}{4}n\big)\big)$ can be similarly constructed. From equation (43), the structure function can be determined:

$$\mathcal{F}\left(\boldsymbol{\alpha}\left(N_0-\frac{1}{4}n\right)\right)=\mathcal{C}_0-\frac{1}{4}\Omega_4\left(\mathcal{C}+\frac{1}{8}(n-1)\delta\right)n + \sum_{s=0}^{3}\left(\left(\mathcal{C}+\frac{1}{8}(2n-1)\delta\right)c_s-\delta d_s\right)\Delta_{s,n}, \tag{55}$$

where $n=0,1,2,\ldots$ and $\mathcal{C}_0$ is chosen to set the structure function positive definite. In the above summation, only the term satisfying the condition $s=n \bmod 4$ contributes to the result. The constants c_s and d_s for $s=0,1,2,3$ are given by

$$\begin{aligned}
c_0&=\tfrac{1}{8}(3\omega_0+\omega_1-\omega_2-3\omega_3), & d_0&=\tfrac{1}{32}(3\omega_0+\omega_1+\omega_2+3\omega_3),\\
c_1&=\tfrac{1}{8}(3\omega_1+\omega_2-\omega_3-3\omega_0), & d_1&=\tfrac{1}{32}(3\omega_1+\omega_2+\omega_3+3\omega_0),\\
c_2&=\tfrac{1}{8}(3\omega_2+\omega_3-\omega_0-3\omega_1), & d_2&=\tfrac{1}{32}(3\omega_2+\omega_3+\omega_0+3\omega_1),\\
c_3&=\tfrac{1}{8}(3\omega_3+\omega_0-\omega_1-3\omega_2), & d_3&=\tfrac{1}{32}(3\omega_3+\omega_0+\omega_1+3\omega_2).
\end{aligned} \tag{56}$$

The energy spectrum for the shape invariant potentials in four steps can be algebraically determined from equations (55) and (17), where 4 replaces k in the latter equation.

4.4. Shape invariance in five steps

Before arriving at the discussion on the shape invariance condition in arbitrary k steps, let us present one more example, that is $k=5$. After this example, we should gather enough information and will be able to deduce the common algebraic structures that share with all the translational shape invariant potentials.

The Z_5-graded Fock space for the shape invariance in five steps is $\mathcal{H}=\sum_{s=0}^{4}\oplus\mathcal{H}_s$. The Fock subspaces are $\mathcal{H}_s=\big\{\big|N_0-n-\frac{s}{5}\big\rangle|n=0,1,2,\ldots\big\}$, for $s=0,1,2,3,4$. The projection operators are given by

$$\begin{aligned}
\Pi_0&=\tfrac{1}{5}(1+2\cos 2\pi\mathcal{N}+2\cos 4\pi\mathcal{N}),\\
\Pi_1&=\tfrac{1}{5}(1-c_2\sin 2\pi\mathcal{N}-c_4\cos 4\pi\mathcal{N}-s_2\sin 2\pi\mathcal{N}+s_4\sin 4\pi\mathcal{N}),\\
\Pi_2&=\tfrac{1}{5}(1-c_4\sin 2\pi\mathcal{N}-c_2\cos 4\pi\mathcal{N}+s_4\sin 2\pi\mathcal{N}+s_2\sin 4\pi\mathcal{N}),\\
\Pi_3&=\tfrac{1}{5}(1-c_4\sin 2\pi\mathcal{N}-c_2\cos 4\pi\mathcal{N}-s_4\sin 2\pi\mathcal{N}-s_2\sin 4\pi\mathcal{N}),\\
\Pi_4&=\tfrac{1}{5}(1-c_2\sin 2\pi\mathcal{N}-c_4\cos 4\pi\mathcal{N}+s_2\sin 2\pi\mathcal{N}-s_4\sin 4\pi\mathcal{N}),
\end{aligned} \tag{57}$$

where $c_2=\frac{1}{2}(1-\sqrt{5})$, $c_4=\frac{1}{2}(1+\sqrt{5})$, $s_2=\sqrt{(5+\sqrt{5})/2}$ and $s_4=-\sqrt{(5-\sqrt{5})/2}$.

When the remainder $R(\boldsymbol{\alpha}(\mathcal{N}))$ in equation (40) acts on the first five eigenstates $|N_0\rangle$, $\big|N_0-\frac{1}{5}\big\rangle$, $\big|N_0-\frac{2}{5}\big\rangle$, $\big|N_0-\frac{3}{5}\big\rangle$ and $\big|N_0-\frac{4}{5}\big\rangle$, the corresponding eigenvalues are $\mathcal{C}\omega_0$, $\big(\mathcal{C}+\frac{1}{5}\delta\big)\omega_1$, $\big(\mathcal{C}+\frac{2}{5}\delta\big)\omega_2$, $\big(\mathcal{C}+\frac{3}{5}\delta\big)\omega_3$ and $\big(\mathcal{C}+\frac{4}{5}\delta\big)\omega_4$, respectively. Other values can be obtained in the same way. The structure function for the shape invariant potentials in five steps is determined by equation (43) as $(n=0,1,2,\ldots)$

$$\mathcal{F}\left(\boldsymbol{\alpha}\left(N_0-\frac{1}{5}n\right)\right)=\mathcal{C}_0-\frac{1}{5}\Omega_5\left(\mathcal{C}+\frac{1}{10}(n-1)\delta\right)n + \sum_{s=0}^{4}\left(\left(\mathcal{C}+\frac{1}{10}(2n-1)\delta\right)c_s-\delta d_s\right)\Delta_{s,n}. \tag{58}$$

Similarly, the term satisfying the condition $s = n \bmod 5$ survives in the above summation. The constant $\mathcal{C}_0$ is chosen to make the associated structure function positive. The constants c_s and d_s for $s = 0, 1, 2, 3, 4$ are given in the following structural patterns:

$$\begin{aligned}
c_0 &= \tfrac{1}{5}(2\omega_0 + \omega_1 - \omega_3 - 2\omega_4), & d_0 &= \tfrac{1}{50}(4\omega_0 + \omega_1 + \omega_3 + 4\omega_4),\\
c_1 &= \tfrac{1}{5}(2\omega_1 + \omega_2 - \omega_4 - 2\omega_0), & d_1 &= \tfrac{1}{50}(4\omega_1 + \omega_2 + \omega_4 + 4\omega_0),\\
c_2 &= \tfrac{1}{5}(2\omega_2 + \omega_3 - \omega_0 - 2\omega_1), & d_2 &= \tfrac{1}{50}(4\omega_2 + \omega_3 + \omega_0 + 4\omega_1),\\
c_3 &= \tfrac{1}{5}(2\omega_3 + \omega_4 - \omega_1 - 2\omega_2), & d_3 &= \tfrac{1}{50}(4\omega_3 + \omega_4 + \omega_1 + 4\omega_2),\\
c_4 &= \tfrac{1}{5}(2\omega_4 + \omega_0 - \omega_2 - 2\omega_3), & d_4 &= \tfrac{1}{50}(4\omega_4 + \omega_0 + \omega_2 + 4\omega_3).
\end{aligned} \tag{59}$$

The energy spectrum for the shape invariant potentials in five steps can be algebraically determined from equations (58) and (17).

4.5. Shape invariance in arbitrary k steps

We are now in a position to deduce the generally algebraic structures for the shape invariant potentials in arbitrary k steps, based on the systematic properties of the previous four examples shown above.

The Z_k-graded Fock space of the shape invariance in k steps is $\mathcal{H} = \sum_{s=0}^{k-1} \oplus \mathcal{H}_s$. The Fock subspaces are denoted by $\mathcal{H}_s = \left\{ \left| N_0 - n - \frac{s}{k} \right\rangle | n = 0, 1, 2, \ldots \right\}$, for $s = 0, 1, \ldots, k-1$. The remainder $R(\boldsymbol{\alpha}(\mathcal{N}))$ (40) acts on the eigenstates $\left| N_0 - \frac{n}{k} \right\rangle$ rendering this result: $\left(\mathcal{C} + \frac{n}{k}\delta\right) \sum_{s=0}^{k-1} \omega_s \Delta_{s,n}$. The structure function for the shape invariant potentials in k steps is again constructed by equation (43) as ($n = 0, 1, 2, \ldots$)

$$\begin{aligned}
\mathcal{F}\left(\boldsymbol{\alpha}\left(N_0 - \frac{n}{k}\right)\right) &= \mathcal{C}_0 - \frac{\Omega_k}{k}\left(\mathcal{C} + \frac{1}{2k}(n-1)\delta\right)n\\
&\quad + \sum_{s=0}^{k-1}\left[\left(\mathcal{C} + \frac{1}{2k}(2n-1)\delta\right)c_s - \delta d_s\right]\Delta_{s,n}.
\end{aligned} \tag{60}$$

In the summation, the $\Delta_{s,n}$ term chooses what satisfies the condition $s = n \bmod k$. $\mathcal{C}_0$ is a constant to render the associated structure function positive. Moreover, the constants c_s and d_s for $s = 0, 1, \ldots, k-1$ are determined based on the structural similarities in the corresponding counterparts of the previous four examples. After some algebra, the general results of c_s and d_s are found to be

$$\begin{aligned}
c_s &= \frac{1}{2k}\sum_{t=0}^{k-1}(k - 1 - 2t)\omega_{s+t},\\
d_s &= \frac{1}{2k^2}\sum_{t=0}^{k-1}(t^2 - (k-1)(t-1))\omega_{s+t},
\end{aligned} \tag{61}$$

where we use the convention $\omega_{s+t} \equiv \omega_{s+t \bmod k}$ (e.g. $\omega_{k+1} = \omega_1$) in these two general formulas. An interesting observation is that if we define $D(t) = t^2 - (k-1)(t-1)$, then $\frac{\mathrm{d}}{\mathrm{d}t}D(t) = -(k - 1 - 2t)$. It is easily checked that the general form in equation (61) exactly reproduces what we have calculated and presented for the shape invariant potentials in the lower steps. This general form is even true for the higher-step shape invariant potentials, for which the detailed results are not shown here.

5. Conclusions

In the present work, we explore the algebraic properties of the shape invariance condition in arbitrary k steps, within the framework of SUSYQM. By imposing extra relations of identification (8) among the k superpotentials and the k remainders, respectively, we obtain the associated simplified potential algebra of shape invariance in k steps (13). This simplified version of the potential algebra is found to be similar to that of shape invariance in one step. Moreover, for translational shape invariance, it is also found to be equivalent to the generalized deformed oscillator algebra, having a built-in Z_k-grading structure. As a result, the algebraic realization of the shape invariance condition in k steps is formulated by the method of the Z_k-graded generalized deformed oscillator. Based on this method, the representation of the simplified potential algebra of shape invariance in k steps can be constructed in terms of the generators of the deformed harmonic oscillator $\{I, \mathcal{A}, \mathcal{A}^\dagger, \mathcal{N}\}$ as well as the grading generator T of the cyclic group of order k.

For the purpose of illustration, we in addition work out four typical examples of shape invariant potentials in $k = 2, 3, 4$ and 5 steps, respectively, in which the parameters a_0 and a_1 of partner potentials are related to each other by $a_1 = a_0 + \delta$. We here assume that the remainder function depends only linearly on the translational parameter by $R_s(a_m) = \sigma_s + a_m\omega_s$ ($s = 0, 2, \ldots, k-1$) as given in equation (18). In each example, the projection operators Π_s are explicitly constructed, which project upon the respective Fock subspace $\mathcal{H}_s$ of the Z_k-graded Fock space $\mathcal{H}$. The associated structure function $\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N}))$ for each example is also established by directly solving equation (43). Next, by inspecting the algebraic structures for these four examples, we learn that the constants c_s and d_s appearing in the structure function seem to follow some particularly systematic rules. Hence, we continue to explore the shape invariance condition in arbitrary k steps and by deduction establish the general algebraic properties for the translational shape invariant potentials. The energy eigenvalues of the initial Hamiltonian of shape invariance in k steps consequently are determined by purely algebraic means.

We conclude the paper by pointing out that the results obtained in this work contain the so-called cyclic shape invariant potentials of period k as a special case. This can be shown by setting $\delta = 0$ in equation (60); then the resultant structure function is nothing but that of the harmonic oscillator potential in k steps, in which the remainder $R\big(\boldsymbol{\alpha}\big(N_0 - \frac{n}{k}\big)\big)$ (up to the multiplicative constant $\mathcal{C}$) is arranged as $\omega_0, \omega_1, \omega_2, \ldots, \omega_{k-1}, \omega_0, \ldots$. The corresponding energy spectrum of the k-step shape invariant Hamiltonian (17) is

$$E_n^{(-)} = \mathcal{C}\left(\frac{n}{k}\Omega_k + c_0 - \sum_{s=0}^{k-1} c_s \Delta_{s,n}\right). \tag{62}$$

Such a potential has been reported in the study of cyclic shape invariant potentials [13, 18, 19]. Besides this, equation (60) contains two new types of shape invariant potentials in k steps, as well. If we set $\delta > 0$ in equation (60), the Pöschl–Teller I potential in k steps results, which possesses an infinite number of bound states. In principle, this potential can be constructed by generalizing the ordinary shape invariant Pöschl–Teller I potential from one step to k steps. Similarly, if $\delta < 0$ is chosen in equation (60), we obtain the Pöschl–Teller II potential in k steps that generalizes the ordinary Pöschl–Teller II potential from shape invariance in one step to k steps, and contains a finite number of bound states. The analytical properties for the Pöschl–Teller I & II potentials in $k > 1$ steps are still unsolved at present, except for the case of $k = 2$.

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